Determination of the shape of the ear channel *†

A.G. Ramm
Mathematics Department, Kansas State University,
Manhattan, KS 66506-2602, USA
ramm@math.ksu.edu

Abstract

It is proved that the measurement of the acoustic pressure on the ear membrane allows one to determine the shape of the ear channel uniquely.

1 Introduction

Consider a bounded domain $D \subset \mathbb{R}^n$, $n = 3$, with a Lipschitz boundary S. Let F be an open subset on S, a membrane, $G = S \setminus F$, $\Gamma = \partial F$, and N is the outer unit normal to S.

Consider the problem:

$$\nabla^2 u + k^2 u = 0 \text{ in } D, \quad u = f \text{ on } F, \quad u = 0 \text{ on } G. \tag{1.1}$$

We assume that k^2 is not a Dirichlet eigenvalue of the Laplacian in D. This assumption will be removed later. If this assumption holds, then the solution to problem (1.1) is unique. Thus, its normal derivative, $h := u_N$ on F, is uniquely determined. Suppose one can measure h on F for some $f \in C^1(F),\ f \not\equiv 0$.

The inverse problem (IP) we are interested in can now be formulated:

Does this datum determine G uniquely?

Thus, we assume that F, f and h are known, that k^2 is not a Dirichlet eigenvalue of the Laplacian in D, and want to determine the unknown part G of the boundary S.

Let Λ be the smallest eigenvalue of the Dirichlet Laplacian L in D. Let us assume that

$$\Lambda > k^2. \tag{1.2}$$

Then, of course, problem (1.1) is uniquely solvable. Assumption (1.2) in our problem is practically not a serious restriction, because the wavelength in our experiment can be

*key words: acoustic waves, inverse problems, ear
†AMS subject classification: 35R30, 74J25, 74J20; PACS 02.30.Jr, 03.40.Kf

chosen as we wish. Since the upper bound on the width d of the ear channel is known, and since

$$\Lambda > \frac{1}{d^2}, \tag{$*$}$$

one can choose $k^2 < \frac{1}{d^2}$ to satisfy assumption (1.2). A proof of the estimate (*) is given at the end of this note.

We discuss the Dirichlet condition but a similar argument is applicable to the Neumann and Robin boundary conditions. Boundary-value problems and scattering problems in rough domains were studied in [1].

Our basic result is the following theorem:

Theorem 1. *If (1.2) holds then the above data determine G uniquely.*

Remark 1. If k^2 is an eigenvalue of the Dirichlet Laplacian L in D, and $m(k)$ is the total multiplicity of the spectrum of L on the semiaxis $\lambda \leq k^2$, then G is uniquely defined by the data $\{f_j, h_j\}_{1\leq j\leq m(k)+1}$, where $\{f_j\}_{1\leq j\leq m(k)+1}$ is an arbitrary fixed linearly independent system of functions in $C(F)$.

In Section 2 proofs are given.

2 Proofs.

Proof of Theorem 1.

Suppose that there are two surfaces G_1 and G_2, which generate the same data, that is, the same function h on F. Let D_1, u_1 and D_2, u_2 be the corresponding domains and solutions to (1.1). Denote $w := u_1 - u_2$, $D^{12} := D_1 \cap D_1$, $D_{12} := D_1 \cup D_2$, $D_3 := D_1 \setminus D^{12}$, $D_4 := D_2 \setminus D^{12}$. Note that $w = w_N = 0$ on F, since the data f and h are the same by our assumption.

Threfore, one has:

$$\nabla^2 w + k^2 w = 0 \text{ in } D^{12}, \quad w = w_N = 0 \text{ on } F \tag{2.1}$$

By the uniqueness of the solution to the Cauchy problem for elliptic equations, one concludes that $w = 0$ in D^{12}. Thus, $u_1 = u_2 = 0$ on ∂D^{12}, and $u_1 = 0$ on ∂D_3. Thus

$$\nabla^2 u_1 + k^2 u_1 = 0 \text{ in } D_3, \quad u_1 = 0 \text{ on } \partial D_3. \tag{2.2}$$

Since $D_3 \subset D$, it follows that $\Lambda(D_3) > \Lambda(D) > k^2$. Therefore k^2 is not a Dirichlet eigenvalue of the Laplacian in D_3, so $u_1 = 0$ in D_3, and, by the unique continuation property, $u_1 = 0$ in D_1. In particular, $u_1 = 0$ on F, which is a contradiction, since $u_1 = f \neq 0$ on F by the assumption. Theorem 1 is proved. $\Box$

Proof of Remark 1. Suppose that $k^2 > 0$ is arbitrarily fixed, and the data are $\{f_j, h_j\}_{1\leq j\leq m(k)}$. Using the same argument as in the proof of Theorem 1, one arrives at the conclusion (2.2) with $u_{j,1}$ in place of u_1, where $u_{j,1}$ solves (1.1) with $f = f_j$, $1 \leq j \leq m(k)+1$. Since the total multiplicity of the spectrum of the Dirichlet Laplacian in D is not more that $m(k)$, one can conclude that $D_1 = D_2$. Remark 1 is proved. $\Box$

We do not discuss in this short note the possible methods for calculating G from the data.

Proof of estimate (*).

Let α be a unit vector, and $d(\alpha)$ be the width of D in the direction α, that is, the distance between two planes, tangent to the boundary S of D and perpendicular to the vector α, so that D lies between these two planes. Let

$$d := \min_{\alpha} d(\alpha) > 0.$$

By the variational definition of Λ one has: $\Lambda = \min \int_D |\nabla u|^2 dx$, where the minimization is taken over all $u \in H^1$, vanishing on S and normalized, $||u||_{L^2(D)} = 1$. Denote $s := x_1$, $y := (x_2, x_3)$, and choose the direction of x_1-axis along the direction α, which minimizes $d(\alpha)$, so that the width of D in the direction of axis x_1 equals d. Then one has:

$$u(s, y) = \int_a^s u_t(t, y)dt,$$

so

$$|u(s, y)|^2 \leq \int_a^s |u_t(t, y)|^2 dt(s - a) \leq d \int_a^b |u_t(t, y)|^2 dt,$$

where $s = a$ and $s = b$ are the equations of the two tangent to S planes, the distance between which is $d = b - a$, and D is located between these planes.

Denote by F_s the crossection of D by the plane $x_1 = s$, $a < s < b$. Integrating the last inequality with respect to y over F_s, and then with respect to s between a and b, one gets:

$$||u||^2_{L^2(D)} \leq d^2 ||\nabla u||^2,$$

which implies inequality (*). □

References

[1] Ramm, A.G., **Inverse Problems**, Springer, New York, 2005.